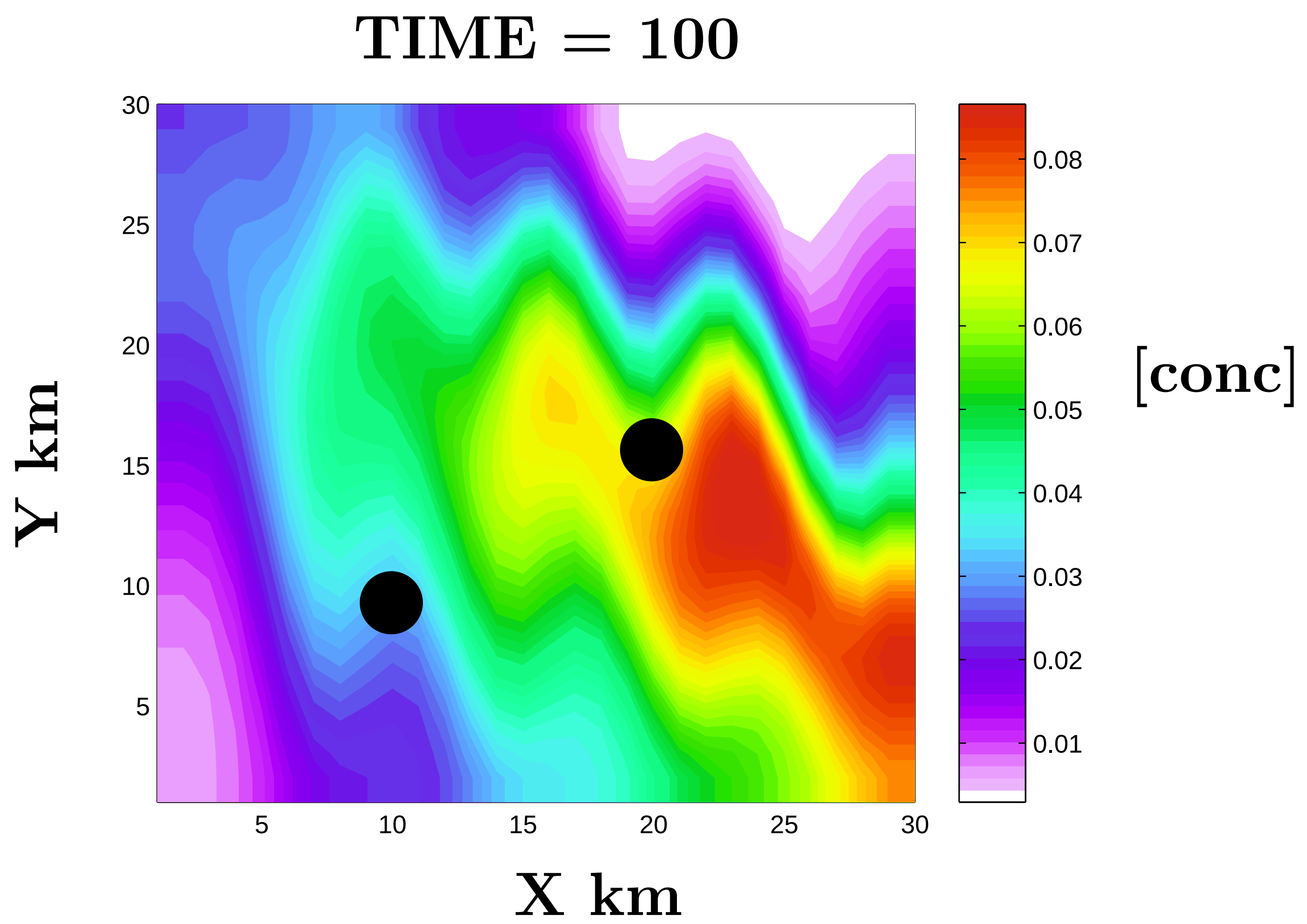
Reconstructing the dispersion of a pollutant



Where are the sources? You only know the solution at time=100

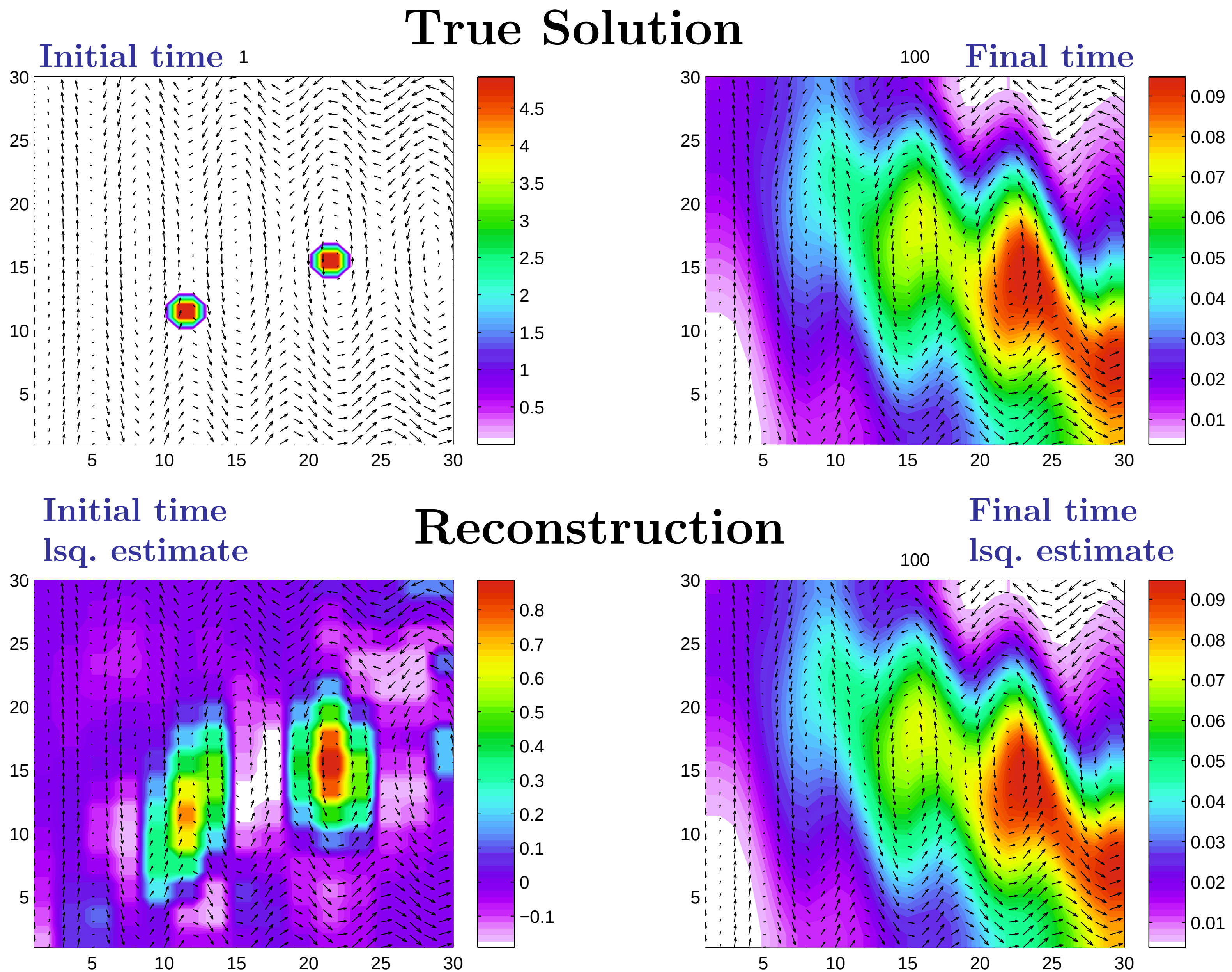
Assume you have a quasi perfect model, where you know diffusion *K*, velocity *u* and *v*

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(1) Least Square Solution

$$\begin{cases} J = \left(\mathrm{y} - \mathrm{E}\mathrm{x}\right)^T \left(\mathrm{y} - \mathrm{E}\mathrm{x}\right) \\ \hat{\mathrm{x}} = \left(\mathrm{E}^T\mathrm{E}\right)^{-1}\mathrm{E}^T\mathrm{y} \end{cases}$$

Where x (the model parameters) are the unkown, y is the values of the tracers at time=100 (which you know) and E is the linear mapping of the initial condition x into y. Matrix E needs to be computed numerically.


True Solution
Initial time
1
Final time
100
Reconstruction
Initial time
lsq. estimate
Final time
lsq. estimate
100

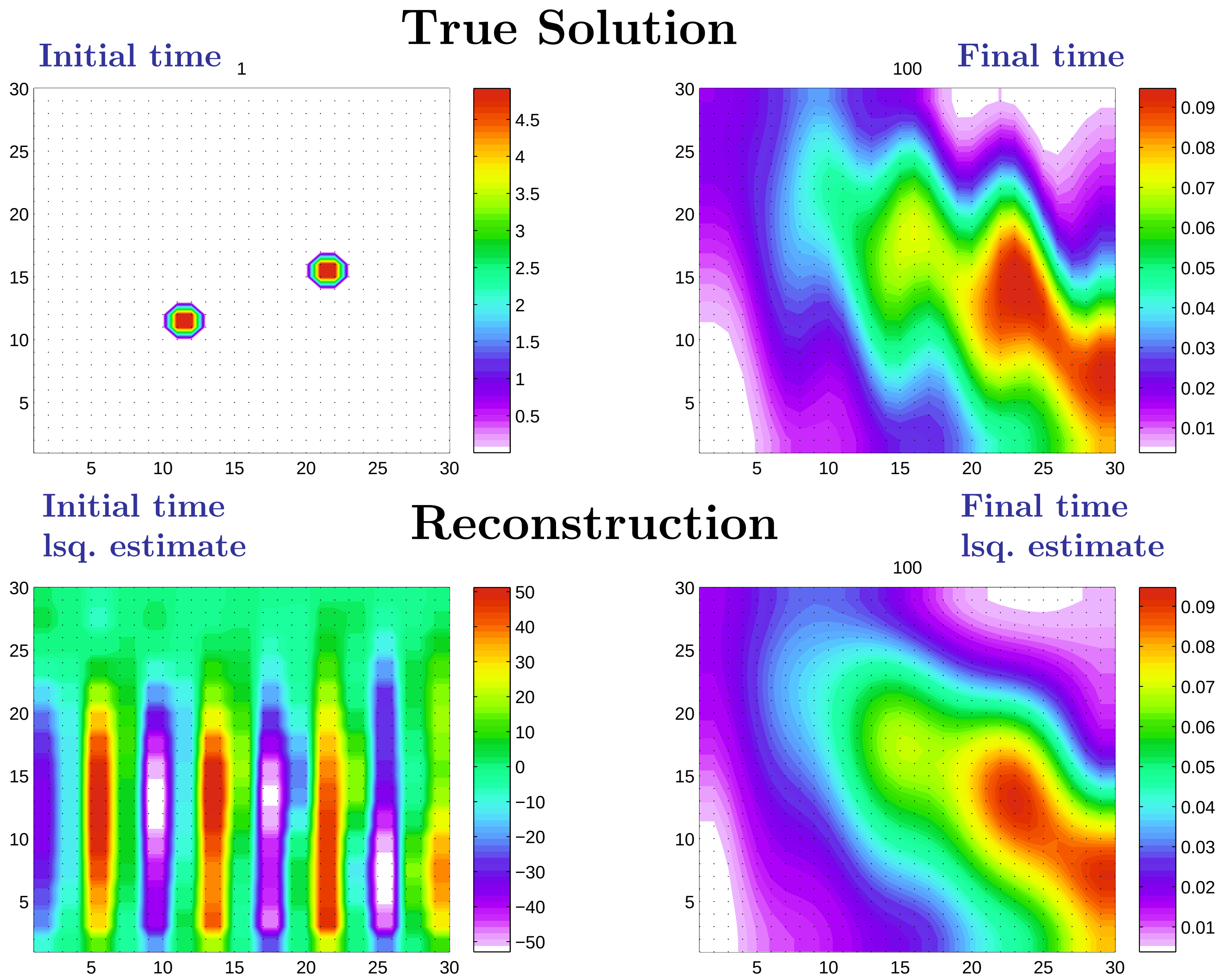
Assume you guess the wrong model.

Say you think there is only diffusion

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(1) Least Square Solution

$$\begin{cases} J = \left(\mathrm{y} - \mathrm{Ex}\right)^T \left(\mathrm{y} - \mathrm{Ex}\right) \\ \hat{\mathrm{x}} = \left(\mathrm{E}^T \mathrm{E}\right)^{-1} \mathrm{E}^T \mathrm{y} \end{cases}$$



Solution looks good at final time, but initial conditions are completely wrong and the values too high

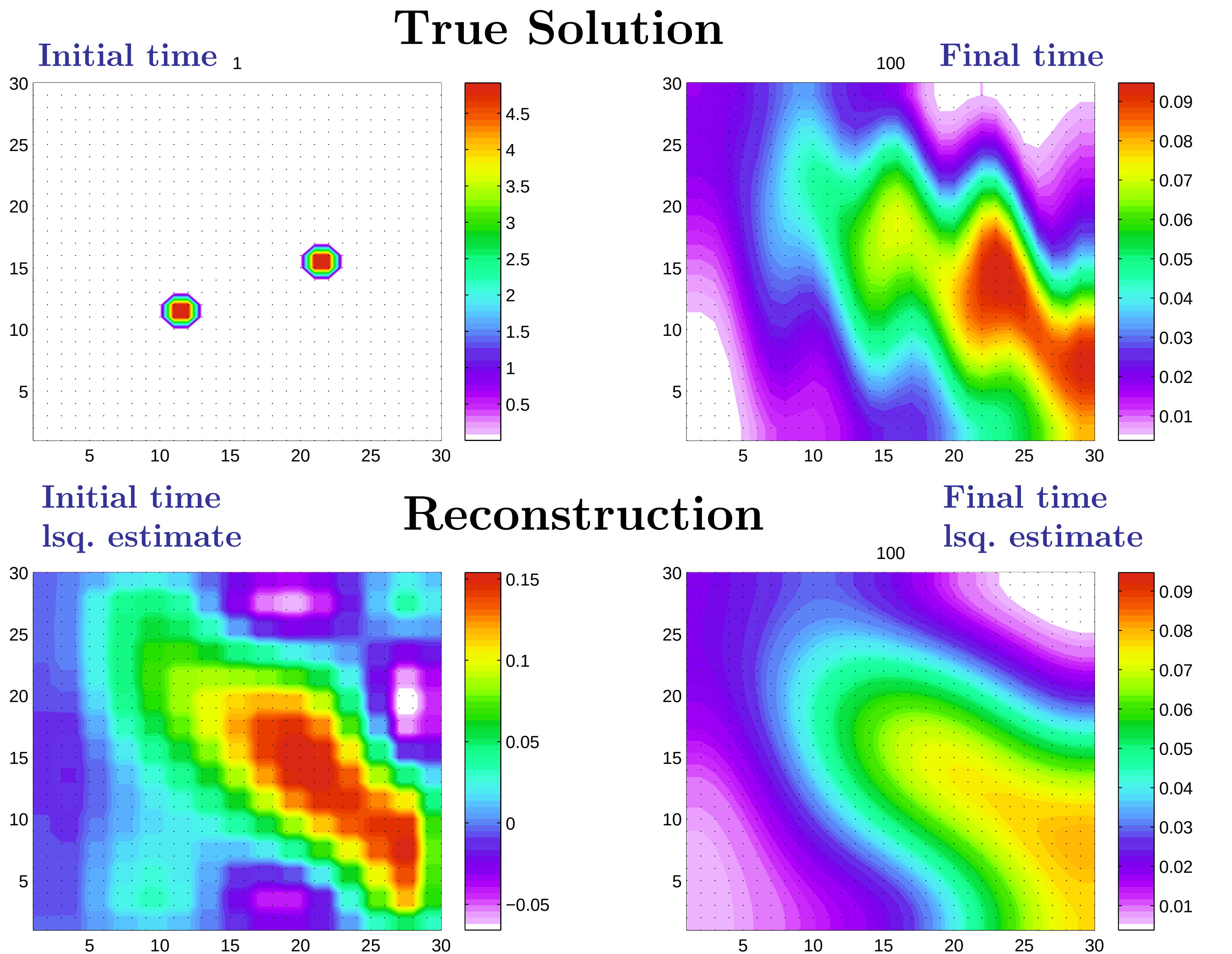
Limit the size of the model parameters!

(which means that the initial condition cannot exceed a certain size)

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(3) Weighted and Tapered Least Square Solution

$$\begin{cases} J = \left(\mathrm{y} - \mathrm{Ex}\right)^T \left(\mathrm{y} - \mathrm{Ex}\right) + \mathrm{x}^T \mathrm{Sx} \\ \hat{\mathrm{x}} = \left(\mathrm{E}^T \mathrm{E} + \mathrm{S}\right)^{-1} \mathrm{E}^T \mathrm{y} \end{cases}$$



Solution looks ok, the initial condition is still unable to isolate the source, given that you have a really bad model not including advection. However the initial condition is reasonable with in the diffusion limit, and the size of the initial condition is also within range.

Say you guess the right model

however velocities are not quite right

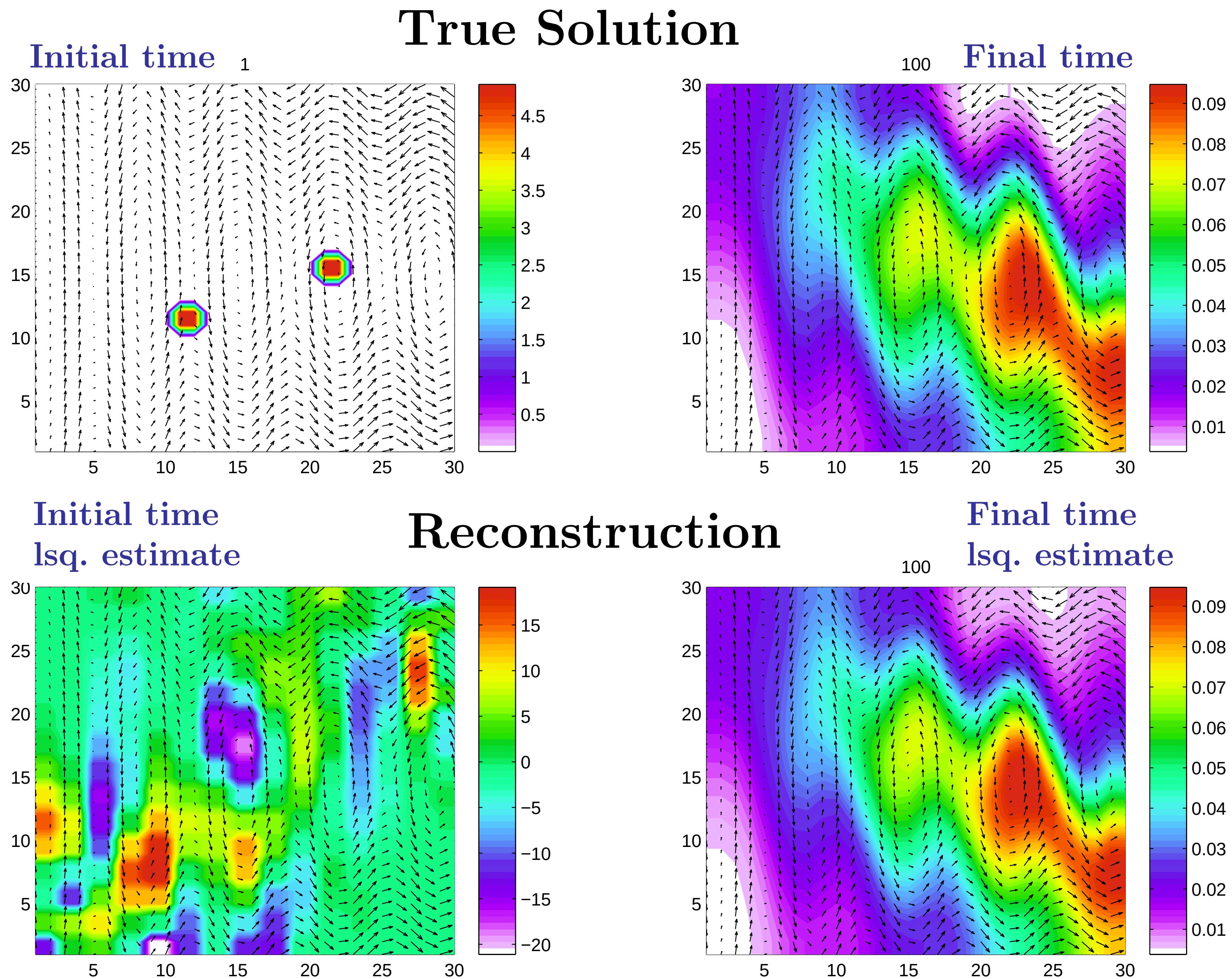
$$\frac{\partial T}{\partial t} + (u + u')\frac{\partial T}{\partial x} + (v + v')\frac{\partial T}{\partial y} = K\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

(u', v') is the error in velocity

Let us try again the strait least square estimate

(1) Least Square Solution

$$\begin{cases} J = (\mathrm{y} - \mathrm{Ex})^T (\mathrm{y} - \mathrm{Ex}) \\ \hat{\mathrm{x}} = (\mathrm{E}^T \mathrm{E})^{-1} \mathrm{E}^T \mathrm{y} \end{cases}$$

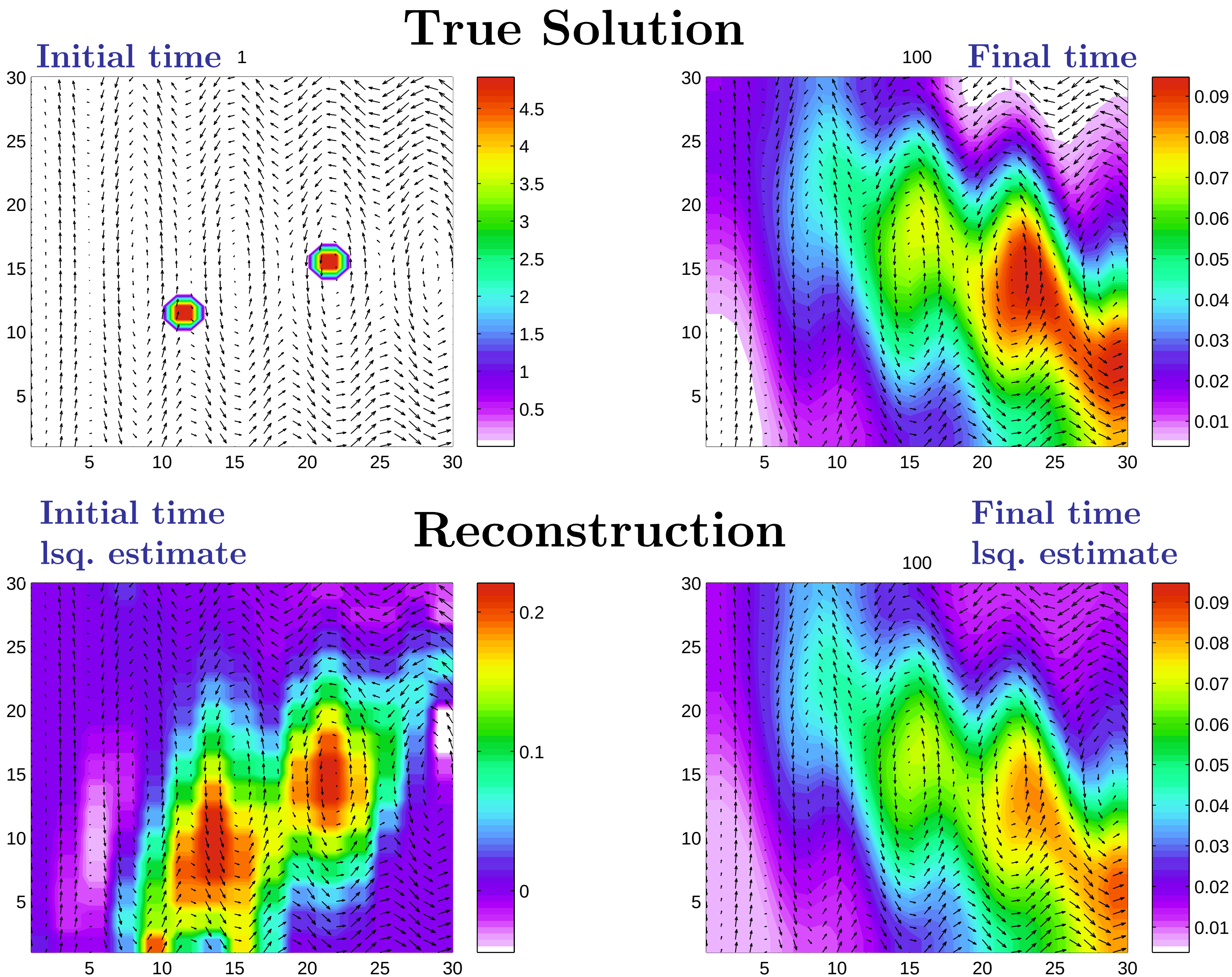


Solution looks great, but again the initial condition totally wrong both in the spatial structure and size. So in this case a small error in our model and too much focus on just fitting the data make the lsq solution useless in terms of isolating the source.

Again limit the size of the model parameters!

(3) Weighted and Tapered Least Square Solution

$$\begin{cases} J = \left(\mathrm{y} - \mathrm{E}\mathrm{x}\right)^{T}\left(\mathrm{y} - \mathrm{E}\mathrm{x}\right) + \mathrm{x}^{T}\mathrm{S}\mathrm{x} \\ \hat{\mathrm{x}} = \left(\mathrm{E}^{T}\mathrm{E} + \mathrm{S}\right)^{-1}\mathrm{E}^{T}\mathrm{y} \end{cases}$$



Solution looks good, the initial condition is able to isolate the sources, the size of the initial condition is within the initial values.

What have we learned?

If you do not have the correct model, it is always a good idea to constrain your model parameters,

you will fit the data less but will have a smoother inversion.