ADVANCED ENVIRONMENTAL DATA ANALYSIS
HOMEWORK #3 Addendum

Using the Adjoint Model to explore sensitivity of the Forward Model dynamics

The goal of this addendum is to understand how we can use the *adjoint* model to explore sensitivity of the final state with respect to changes in initial condition. We first need to define what aspect of the final state we are interested in. For example I may want to know how the variance at location $T_{100}(loc=2)$ changes with respect to changes in the initial condition. Assume this is the location of Atlanta. To do so we define a cost function that measures the variance over Atlanta of our pollutant:

$$J = \mathbf{y}^T \mathbf{W} \mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} T_{100}(loc=1) \\ T_{100}(loc=2) \\ \ldots \\ T_{100}(loc=N) \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ \ldots \\ 0 \end{bmatrix} \qquad \mathbf{W} = \mathbf{w}\mathbf{w}^T$$

The matrix $\mathbf{W}$ defines a set of weights so that we measure the variance only over Atlanta location (in this case location 2) but not at all other locations.

$$\delta J = \delta \mathbf{y}^T \frac{\partial J}{\partial \mathbf{y}}$$

$$\delta \mathbf{y} = \mathbf{E} \delta \mathbf{y_0} \quad \text{and} \quad \frac{\partial J}{\partial \mathbf{y}} = 2\left(\mathbf{y}^T \mathbf{W}\right)^T$$

$$\delta J = \left(\mathbf{E} \delta \mathbf{y_0}\right)^T 2\left(\mathbf{y}^T \mathbf{W}\right)^T = \delta \mathbf{y_0}^T 2 \underbrace{\mathbf{E}^T}_{\text{Adjoint}} \mathbf{W}\mathbf{y}$$

$$\frac{\delta J}{\delta \mathbf{y_0}} = 2 \underbrace{\mathbf{E}^T}_{\text{Adjoint}} \mathbf{W}\mathbf{y}$$

This results shows that changes in the cost function with respect to the initial conditions can be explored using the adjoint model, in that the adjoint model forced with $\mathbf{W}\mathbf{y}$ gives us the gradient of the cost function with respect to $\mathbf{y_0}$

Question:

1) You have 1 PPM of passive tracer that you can spread at your initial condition as you like. Find the initial pattern of this passive $\hat{\mathbf{y}}_0$ tracer so that you maximize changes a t=100 over Atlanta. Print out the value of the cost function J. For the purpose of this exercise place Atlanta at location $w(110)=1$ and enforce that your initial pattern has $std(\hat{\mathbf{y}}_0)=1$ PPM

2) To make sure you found the optimal pattern $\hat{\mathbf{y}}_0$, compare the value you got for J with the ones obtained by randomly perturbing your initial condition $\mathbf{y}_0$ with a pattern that has $std(\mathbf{y}_0)=1$ PPM. Can you find a better pattern that leads to bigger changes in J? You can also try to explore the value of the cost function by perturbing each single location separately.